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A UNIFIED APPROACH TO THE HELIOSEISMIC FORWARD AND INVERSE PROBLEMS OF DIFFERENTIAL ROTATION

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ABSTRACT

We present a general, degenerate perturbation theoretic treatment of the helioseismic forward and inverse problems for solar differential rotation. Our approach differs from previous work in two principal ways. First, in the forward problem, we represent differential rotation as the axisymmetric component of a general toroidal flow field using vector spherical harmonics. The choice of these basis functions for differential rotation over previously chosen ad hoc basis functions (e.g., trigonometric functions or Legendre functions) allows the solution to the forward problem to be written in an exceedingly simple form (eqs. [32]–[37]). More significantly, their use results in inverse problems for the set of radially dependent vector spherical harmonic expansion coefficients, which represent rotational velocity, that decouple so that each degree of differential rotation can be estimated independently from all other degrees (eqs. [56] & [61]–[63]). Second, for use in the inverse problem, we express the splitting caused by differential rotation as an expansion in a set of orthonormal polynomials that are intimately related to the solution of the forward problem (eqs. [5] and [54]). The orthonormal polynomials are Clebsch-Gordon coefficients and the estimated expansion coefficients are called splitting coefficients. The representation of splitting with Clebsch-Gordon coefficients rather than the commonly used Legendre polynomials results in an inverse problem in which each degree of differential rotation is related to a single splitting coefficient (eq. [56]). The combined use of the vector spherical harmonics as basis functions for differential rotation and the Clebsch-Gordon coefficients to represent splitting provides a unified approach to the forward and inverse problems of differential rotation which will greatly simplify inversion. We submit that the mathematical and computational simplicity of both the forward and inverse problems afforded by our approach argues persuasively that helioseismological investigations would be well served if the current ad hoc means of representing differential rotation and splitting would be replaced with the unified methods presented in this paper.

Subject headings: Sun: oscillations — Sun: rotation

1. INTRODUCTION

An acoustic mode of oscillation of a spherically symmetric, nonrotating, adiabatic, static solar model without magnetic fields is typically identified by a trio of quantum numbers that represents its displacement field: n, the radial order; l, the spherical harmonic degree; and m, the azimuthal order of the mode. Because of the rotational symmetries of this model, the modes of oscillation are $2l + 1$ degenerate. That is, the frequencies of the $2l + 1$ modes with different m values but with the same n and l values are identical. These modes are said to form a multiplet. The real Sun is not so simple. Of particular relevance for this paper is the fact that the Sun is rotating and is deforming internally so that, for example, the surface rate of rotation at the solar equator is greater than at the poles. This phenomenon is known as differential rotation.

A number of ways have been chosen to represent differential rotation mathematically. We will argue in this paper that a representation with exceptionally nice consequences is the solar rotational velocity $\boldsymbol{v}_{\rm rot}(r, \theta, \phi)$, defined to be the axisymmetric component of general toroidal flow fields in the solar interior. A heretofore more popular, and perhaps more conceptually appealing, way of looking at this is that the Sun is rotating differentially and that the rotation rate $\Omega(r, \theta)$ is itself a function of both radial position and colatitude. Rotational velocity $\boldsymbol{v}_{\rm rot}(r, \theta, \phi)$ and rotation rate $\Omega(r, \theta)$ are simply related by

$$\boldsymbol{v}_{\rm rot}(r, \theta, \phi) = \boldsymbol{\Omega}(r, \theta, \phi) \times \boldsymbol{r} = \Omega(r, \theta) r \sin(\theta) \hat{\boldsymbol{\phi}} \qquad (1)$$

where $\boldsymbol{\Omega} = \Omega \hat{z}$, $\hat{z}$ is the unit vector which points along the axis of rotation, and $\boldsymbol{r}$ is the position vector from the center of the Sun to position (r, θ, ϕ). The coordinates $\boldsymbol{r} = (r, \theta, \phi)$ are spherical polar coordinates (where θ is colatitude) and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ denote unit vectors in the coordinate directions. However represented, solar differential rotation lifts the degeneracy of the solar acoustic or p-modes, splitting the frequencies of oscillation of the Sun. This phenomenon is, without doubt, the largest contributor to the splitting of solar oscillations.

Observations of split solar p-mode frequencies date from Claverie et al. (1981). The quantity and quality of new measurements have been steadily improving (e.g., Gough 1982; Hill, Bos, & Goode 1982; Duvall & Harvey 1984; Duvall, Harvey, & Pomerantz

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1986; Brown 1985; Libbrecht 1986, 1989; Brown & Morrow 1987; Tomczyk 1988; Rhodes et al. 1990). Since p-modes are dominantly split by differential rotation, a first-order understanding of the data requires the estimation of models of rotation that predict the data accurately. Furthermore, an understanding of the solar angular momentum budget requires accurate models of differential rotation (Gilman, Morrow, & DeLuca 1989). For these reasons, the inverse problem of inferring the radial and latitudinal dependence of differential rotation from observed split frequencies has been a major focus of helioseismology. The reader is referred to Brown et al. (1989), Christensen-Dalsgaard, Schou, & Thompson (1990), and Thompson (1990) for clear discussions of the current state of this venture.

A necessary preface to tackling the inverse problem is the solution of the forward problem: i.e., given a model of solar differential rotation, determine the split p-mode frequencies. This problem was first solved by Cowling & Newing (1949) and subsequent treatments can be found in Ledoux (1951), Hansen, Cox, & Van Horn (1977), Gough (1981), and Brown (1985). Naturally, the form of the solution of the forward problem is a function of the basis functions chosen to represent differential rotation. Since differential rotation is axisymmetric and even about the equatorial plane, it admits a very simple mathematical representation.

In helioseismology, there are two popular ways chosen to represent differential rotation, both of which are based on polynomial expansions of rotation rate Ω rather than rotational velocity $\boldsymbol{v}_{\rm rot}$. The more popular of these parameterizations is to expand $\Omega(r, \theta)$ in even powers of $\cos\theta$ (see Brown et al. 1989):

$$\Omega(r, \theta) = \sum_{k=0,2,4,\ldots}^{\infty} \bar{\Omega}_k(r) \cos^k \theta \tag{2}$$

where $\bar{\Omega}_0(r)$ describes the bulk rotation rate of the Sun and the $\bar{\Omega}_k(r)$ for $k > 0$ describe the radial dependence of latitudinally dependent differential rotation. The popularity of this representation probably derives from tradition, since it was used in early studies of differential rotation made from observations of the solar surface (e.g., Howard & Harvey 1970). Thus, the use of these basis functions eased comparison with direct observations of differential rotation. It is generally recognized that a problem with this parameterization is that the basis functions are not orthogonal. This is not really an obstacle from a forward theoretic perspective, but is troublesome inverse theoretically since future inversions for higher degree components must also redo the lower degree components since, strictly speaking, they are not independent of one another. The use of any set of orthogonal basis functions overcomes this problem. In particular, Legendre polynomials in cosine colatitude have seen some application (e.g., Korzennik et al. 1988):

$$\Omega(r, \theta) = \sum_{k=0,2,4,\ldots}^{\infty} \Omega_k(r) P_k(\cos\theta)\ . \tag{3}$$

Although the parameterizations of differential rotation given by equations (2) and (3) are intuitively simple and allow straightforward comparison to other kinds of observations, their use leads to unfortunate consequences that can be entirely circumvented with a more judicious choice of basis functions. Problems with these basis functions include the following. (1) They do not generalize easily to general nonaxisymmetric flows. (2) They do not yield conveniently to the elegant generalized spherical harmonic formalism of Phinney & Burridge (1973) and are, therefore, computationally cumbersome. There is a more significant problem arising from the way observers choose to represent splitting data as a Legendre polynomial expansion in (m/l):

$$\omega_{nl}^m = \omega_{nl} + l \sum_{i=1}^{M} {}_n a_{li} P_i(m/l)\ . \tag{4}$$

The expansion coefficients ${}_n a_{li}$ are commonly called splitting coefficients. The use of equations (2) or (3) as basis functions for differential rotation applied to splitting data represented with equation (4) leads to a serious practical problem troubling inversion; namely, (3) they generate a coupled set of inverse problems in which for a given k the estimation of $\Omega_k(r)$ or $\bar{\Omega}_k(r)$ depends on $\Omega_{k'}(r)$ or $\bar{\Omega}_{k'}(r)$, respectively, for all even $k' > k$.

Problem 3 has been tackled in a number of ways, including: (a) by estimating $\Omega_k(r)$ or $\bar{\Omega}_k(r)$ for all even $k \leq k_{\rm max}$ simultaneously, where $k_{\rm max} = 4$ usually (e.g., Thompson 1990); (b) by estimating each $\Omega_k(r)$ or $\bar{\Omega}_k(r)$ recursively by solving first for each even k' where $k_{\rm max} > k' > k$ (e.g., Brown et al. 1989); (c) by forming recombined basis functions that allow, in the high l limit, the inverse problems for distinct k to decouple (e.g., Korzennik et al. 1988); and (d) by replacing equation (4) with an alternative representation of splitting measurements relative to which recombined basis functions decouple in the inverse problems for distinct k (Durney, Hill, & Goode 1988). There are problems with each of these approaches. Approaches a and b generate models that at different degrees k have correlated errors. In addition, approach b necessarily performs the recursion in the direction opposite from how a stable and robust recursive technique should be applied. A robust recursive technique would first estimate the features of the model that have the largest expression in the data. In this case, these are the longest wavelength features of the model (i.e., small k). Then these should be used in the estimation of shorter wavelength model components (i.e., higher k) that affect the data more subtly. Approach b does the opposite of this. By estimating the shorter wavelength features first, it propagates errors from the more poorly constrained to the better constrained features of the model. Approach c has a limited range of applicability and approach d requires observers to summarize their data in a form they consider suboptimal.

The common problem with all of these approaches to problem 3 is that the basis functions that have been chosen to represent both splitting and differential rotation have been ad hoc. In this paper we take a different approach. We define the inverse problem explicitly in terms of the form of the solution to the forward problem. We show that there exists a natural set of basis functions with which to represent differential rotation such that the inverse problems for different degrees of differential rotation decouple with respect to data represented in the usual way (eq. [4]). Consequently, joint and recursive inversions as well as asymptotic approximations can be entirely circumvented. These basis functions are simply the vector spherical harmonic components of $\boldsymbol{v}_{\rm rot}$. In addition, vector spherical harmonics generalize easily to nonaxisymmetric flow fields and yield to the formalism of Phinney & Burridge; thus, their use also addresses problems (1) and (2). We also show that the inverse problem is simplified further if splitting data are expressed using a set of natural basis functions intimately related to the solution of the forward problem. These orthonormal functions are the Clebsch-Gordon coefficients β_{il}^{m} with which the $2l+1$ frequencies of a single split multiplet would be represented as follows:

$$\omega_{nl}^{m} = \omega_{nl} + \sum_{i=1}^{M} {}_{n}b_{li}\beta_{il}^{m} . \tag{5}$$

The expansion coefficients ${}_{n}b_{li}$ represent a set of new splitting coefficients.

Our approach to the forward problem is motivated by the approach geophysicists have taken to determine the splitting and coupling of terrestrial oscillations caused by aspherical perturbations in the elastic moduli and density of the Earth (e.g., Dahlen 1968, 1969; Luh 1973, 1974; Woodhouse & Dahlen 1978; Woodhouse 1980; Woodhouse & Girnius 1982). An approximation that has proven useful in terrestrial applications is to allow modes to couple only if they share the same radial order n and harmonic degree l. This means that if two modes are not degenerate in the absence of asphericities, they will not be considered potential coupling partners in the presence of the asphericity. In this case, it is appropriate to use degenerate perturbation theory to compute the split frequencies. This approximation is known to geophysicists as the isolated multiplet approximation since it is accurate if the $2l+1$ modes composing a multiplet are isolated in complex frequency from modes composing other multiplets. In terrestrial applications this approximation has proven to be highly useful and quite accurate for calculating split frequencies but is not as useful for computing modal displacements. To compute modal displacements accurately, quasi-degenerate perturbation theory has been used by geophysicists in which modes are allowed to couple even if they are only nearly degenerate with respect to the spherical Earth model. For the Sun, the number of significant accidental near degeneracies between modes from different multiplets satisfying the selection rules that govern coupling for differential rotation is vanishingly small. Thus, solar multiplets can accurately be considered isolated in complex frequency and degenerate perturbation theory can be used to compute the split frequencies. The solution to the forward problem we present is accurate to first order in Ω/ω in the absence of accidental degeneracies, where Ω is the differential rotational frequency and ω is a modal frequency. Due to the spacing of p-modes between and along dispersion branches, the contribution to splitting caused by quasi-degenerate coupling between modes from different multiplets is an effect of higher order than first in Ω/ω. We will discuss this further in a future contribution.

In § 2, we defined the new basis functions for differential rotation mathematically and relate them to the previously used basis functions given by equations (2) and (3). In § 3, we present the solution to the forward problem for differential rotation using degenerate perturbation theory. The solution is expressed in terms of Wigner 3-j symbols which are straightforward to compute numerically. For low-degree differential rotation, we present analytical expressions for the Wigner 3-j symbols in terms of simple polynomials in m and l. In § 4, we discuss the use of Clebsch-Gordon coefficients for representing splitting data, by using equation (5) as an alternative to equation (4). These coefficients form an orthonormal basis set and are simply related to the Wigner 3-j symbols found in the solution of the forward problem. The use of the Clebsch-Gordon coefficients to represent splitting provides a unified approach to the data analysis and inverse problems. In § 5, we derive the form of the inverse problems relating the basis functions for $\boldsymbol{v}_{\rm rot}$ both to the new splitting coefficients b_i and to the traditional splitting coefficients a_i. In both formulations, a single degree of the vector spherical harmonic expansion of rotational velocity can be related to a linear combination of the splitting coefficients. However, by using the Clebsch-Gordon coefficients, the sum is particularly simple, reducing to a single term. For completeness, we present formulae for converting solutions back to the rotation rate basis functions for differential rotation. Finally, to simplify the recommended use of the Clebsch-Gordon coefficients β_{il}^{m} for representing splitting, algebraic expressions in m and l for low-degree i coefficients are presented in the Appendix.

2. VECTOR SPHERICAL HARMONIC REPRESENTATION OF DIFFERENTIAL ROTATION

It is useful to decompose a general stationary, laminar velocity field $\boldsymbol{v}(r, \theta, \phi)$ into poloidal $\boldsymbol{P}$ and toroidal $\boldsymbol{T}$ components:

$$\boldsymbol{v}(r, \theta, \phi) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} [\boldsymbol{P}_s^t(r, \theta, \phi) + \boldsymbol{T}_s^t(r, \theta, \phi)] . \tag{6}$$

The poloidal and toroidal components can be fully characterized by the radius dependent vector spherical harmonic expansion coefficients $u_s^t(r)$, $v_s^t(r)$, and $w_s^t(r)$:

$$\boldsymbol{P}_s^t(r, \theta, \phi) = u_s^t(r) Y_s^t(\theta, \phi)\hat{\boldsymbol{r}} + v_s^t(r)\nabla_1 Y_s^t(\theta, \phi) , \tag{7}$$

$$\boldsymbol{T}_s^t(r, \theta, \phi) = - w_s^t(r)\hat{\boldsymbol{r}} \times \nabla_1 Y_s^t(\theta, \phi) , \tag{8}$$

where the surface gradient operator, ∇_1, is given by

$$\nabla_1 = r[\nabla - \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \nabla)] = \hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{\sin \theta} \frac{\partial}{\partial \phi} . \tag{9}$$

The function $Y_s^t(\theta, \phi)$ is a spherical harmonic of degree s and azimuthal order t defined using the convention of Edmonds (1960) as

$$Y_s^t = (-1)^t \left[\frac{2s+1}{4\pi} \frac{(t-s)!}{(t+s)!}\right]^{1/2} P_s^t(\cos \theta) e^{it\phi} , \tag{10}$$

where $Y_s^{-t} = (-1)^t[Y_s^{-t}]^*$, P_s^t are associated Legendre functions, and the asterisk represents complex conjugation. The normalization constants in equation (10) have been chosen such that

$$\int_0^{2\pi}\int_0^{\pi}[Y_{s'}^{t'}(\theta,\phi)]^* Y_s^t(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_{t't}\delta_{s's}\,, \tag{11}$$

where integration is over the unit sphere. The poloidal coefficients $u_s^t(r)$ and $v_s^t(r)$ are not independent if the anelastic condition $[\nabla \cdot (\rho v) = 0]$ is imposed.

The differential rotation velocity field $v_{\rm rot}(r, \theta)$ simply corresponds to the odd-degree, zonal part of the toroidal flow field in equation (8) which can be written:

$$v_{\rm rot}(r,\theta) = -\sum_{s=1,3,5,\dots}^{\infty} w_s^0(r)\partial_\theta Y_s^0\hat{\phi}\,, \tag{12}$$

where, for example:

$$\partial_\theta Y_1^0 = -\frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2}\sin\theta\,, \tag{13}$$

$$\partial_\theta Y_3^0 = -\frac{3}{4}\left(\frac{7}{\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)\,, \tag{14}$$

$$\partial_\theta Y_5^0 = -\frac{15}{16}\left(\frac{11}{\pi}\right)^{1/2}\sin\theta(21\cos^4\theta - 14\cos^2\theta + 1)\,. \tag{15}$$

The relationship between the expansion coefficients $w_s^0(r)$ with $\bar{\Omega}_k(r)$ and $\Omega_k(r)$ can be determined by equating the representations in equations (1) and (12), expanding each side in terms of irreducible trigonometric functions, and equating the appropropriate groups of terms. This procedure yields

$$w_1^0(r) = 2\sqrt{\frac{\pi}{3}}\,r\left[\bar{\Omega}_0(r) + \frac{1}{5}\bar{\Omega}_2(r) + \frac{3}{35}\bar{\Omega}_4(r)\right] = 2\sqrt{\frac{\pi}{3}}\,r\left[\Omega_0(r) - \frac{1}{5}\Omega_2(r)\right], \tag{16}$$

$$w_3^0(r) = 2\sqrt{\frac{\pi}{7}}\,r\left[\frac{2}{15}\bar{\Omega}_2(r) + \frac{28}{315}\bar{\Omega}_4(r)\right] = 2\sqrt{\frac{\pi}{7}}\,r\left[\frac{1}{5}\Omega_2(r) - \frac{1}{9}\Omega_4(r)\right], \tag{17}$$

$$w_5^0(r) = 2\sqrt{\frac{\pi}{11}}\,r\left[\frac{8}{315}\bar{\Omega}_4(r)\right] = 2\sqrt{\frac{\pi}{11}}\,r\left[\frac{1}{9}\Omega_4(r)\right], \tag{18}$$

where we have truncated the sums at $k = 4$ in equations (2) and (3) and at $s = 5$ in equation (12).

3. THE FORWARD PROBLEM

The equations governing the effect of differential rotation on solar oscillation frequencies can be presented naturally in either of two reference frames: a frame corotating with the average angular velocity of the Sun or an inertial frame roughly identifiable with the frame of observation. Since differential rotation is stationary relative to both the corotating and inertial frames, solutions to these equations will separate in both frames. For simplicity of use by observers, we choose the inertial frame in which to represent and solve the following equations. As a consequence, $v_{\rm rot}$ will be considered to include the average rotational velocity of the Sun.

Let $k = (n, l)$; then the displacement field for p-modes in the presence of an axisymmetric flow field separates spatially and temporally:

$$s(r, t) = s_k^m(r)e^{i\omega_k^m t}\,. \tag{19}$$

We have chosen to introduce an unfortunate notational conflict and let t represent time (as well as the azimuthal order of a convective flow field). The radial eigenfunctions are defined as follows:

$$s_k^m(r) = {}_nU_l(r)Y_l^m(\theta,\phi)\hat{r} + {}_nV_l(r)\nabla_1 Y_l^m(\theta,\phi)\,, \tag{20}$$

where ${}_nU_l(r)$ and ${}_nV_l(r)$ denote, respectively, the radial and horizontal eigenfunctions for harmonic degree l and radial order n. Hereafter, we drop the subscripts n and l in equation (20) and use instead $U = {}_nU_l(r)$, and $V = {}_nV_l(r)$. The eigenfunctions satisfy an orthogonality condition given by

$$\int \rho_0\, s_{k'}^{m'*}\cdot s_k^m\, d^3r = N\delta_{m'm}\delta_{n'n}\delta_{l'l}\,, \tag{21}$$

where ρ_0 is the density of the equilibrium solar model, and

$$N = \int_0^{R_\odot}\rho_0(U^2 + L^2V^2)r^2\,dr\,, \tag{22}$$

where $L^2 = l(l+1)$. The scalar constant N depends on the normalization of the eigenfunctions U and V.

We seek equations governing the influence of differential rotation on p-mode frequencies. The equation of motion for the oscillations of a spherically symmetric, nonrotating, adiabatic, static solar model without magnetic fields is given by

$$\mathscr{L}(s_k) - \rho_0 \partial_t^2 s_k = 0 , \tag{23}$$

where $\mathscr{L}$ is a linear, self-adjoint spatial integro-differential operator subject to certain idealized boundary conditions. This equation can be rewritten using equation (19) as

$$\mathscr{L}(s_k) + \rho_0 \omega_k^2 s_k = 0 . \tag{24}$$

If we perturb the above described model by adding a rotational velocity field $\boldsymbol{v}_{\rm rot}$, the equation of motion is altered. In particular, the local time derivative ∂_t must be generalized to the material time derivative D_t and the displacements and frequencies of the modes must be perturbed. Thus, we must make the substitutions

$$\partial_t \to D_t = \partial_t + \boldsymbol{v}_{\rm rot} \cdot \nabla , \tag{25}$$

$$s_k \to s_k + \delta s_k , \tag{26}$$

$$\omega_k \to \omega_k + \delta\omega , \tag{27}$$

where ω_k is the degenerate frequency of the unperturbed multiplet. Lynden-Bell & Ostriker (1967) showed that the advection of the velocity field $\boldsymbol{v}_{\rm rot}$ by the displacement eigenfunctions s_k can be ignored.

Making the above substitutions into equation (23), retaining only terms first-order in $\boldsymbol{v}_{\rm rot}$, $\delta\omega$, and δs_k, and using equation (24) to eliminate terms, we obtain the perturbed equation of motion

$$\mathscr{L}(\delta s_k) - 2i\omega_k \rho_0 \boldsymbol{v}_{\rm rot} \cdot \nabla s_k + \rho_0 \omega_k^2 \delta s_k + \rho_0 \delta\omega^2 s_k = 0 . \tag{28}$$

In accordance with our discussion in § 1, we will assume the isolated multiplet approximation in the remainder of this paper and apply degenerate perturbation theory to calculate the split frequencies due to differential rotation. Under this approximation, we expand the perturbed displacement field s_k in terms of the $(2l + 1)$ eigenfunctions of a single multiplet of a spherically symmetric solar model as follows

$$s_k = \sum_{m=-l}^{l} a_m s_k^m(r) . \tag{29}$$

Inserting equation (29) into equation (28), taking the inner product of the resulting expression with $s_k^{m'*}$, integrating over the volume of the Sun, and using equation (24) again, we obtain

$$\int s_k^{m'*} \cdot \mathscr{L}(\delta s_k) d^3r - \int \mathscr{L}(s_k^{m'*}) \cdot \delta s_k \, d^3r + \sum_{m=-l}^{l} a_m \left\{ \delta\omega^2 \int \rho_0 s_k^{m'*} \cdot s_k^m \, d^3r - 2i\omega_k \int \rho_0 s_k^{m'*} \cdot \boldsymbol{v}_{\rm rot} \cdot \nabla s_k^m \, d^3r \right\} = 0 . \tag{30}$$

The first two terms in equation (30) cancel since for an adiabatic solar model $\mathscr{L}$ is self-adjoint. Using this fact, together with the orthonormality of the eigenfunctions given by equation (21), we obtain from equation (30) the shift in squared frequency caused by differential rotation:

$$N \, \delta\omega_m^2 = 2i\omega_k \int \rho_0 s_k^{m*} \cdot \boldsymbol{v}_{\rm rot} \cdot \nabla s_k^m \, d^3r , \tag{31}$$

where we have used the fact that the right-hand side of equation (31) vanishes unless $m' = m$ since $\boldsymbol{v}_{\rm rot}$ is axisymmetric.

Substituting into equation (31) from equations (20) and (12) and using the fact that $\delta\omega^2 \cong 2\omega_k \delta\omega$, we find that for $-l \le m \le l$:

$$\omega_k^m = \omega_k + \delta\omega_m = \omega_{nl} + \sum_{s=1,3,5,\ldots} \gamma_{sl}^m \, {}_n c_{sl} , \tag{32}$$

where

$$\gamma_{sl}^m = (2l+1)L^2 \left(\frac{2s+1}{4\pi} \right)^{1/2} H_s^1 H_s^m F_s^2 , \tag{33}$$

$${}_n c_{sl} = \int_0^{R_\odot} w_s^0(r) \, {}_n K_{ls}(r) r^2 \, dr , \tag{34}$$

$${}_n K_{ls}(r) = -\rho_0 r^{-1} \{ U^2 + L^2 V^2 - [2UV + \tfrac{1}{2} s(s+1) V^2] \} / N , \tag{35}$$

and where we have defined

$$H_s^m F_s = (-1)^{(l-m)} \begin{pmatrix} s & l & l \\ 0 & m & -m \end{pmatrix} , \tag{36}$$

$$F_s = \left[\frac{(2l-s)!}{(2l+s+1)!} \right]^{1/2} . \tag{37}$$

The gradient operator in equation (31) acts on both the scalar components and unit vectors of s_k^m which yield, respectively, what might loosely be called the advection $(U^2 + L^2 V^2)$ and Coriolis $[2UV + \frac{1}{2} s(s+1) V^2]$ contributions to the integral kernel ${}_n K_{ls}$.

The derivation of equation (33) is greatly simplified by use of the generalized spherical harmonic formalism of Phinney & Burridge (1973). It is beyond the scope of this paper to describe their formalism in any detail. However, Lavely & Ritzwoller (1991) do describe the formalism as it applies to the problem of splitting caused by a general convective flow field of which, of course, differential rotation is simply a special case. The form of equation (33), written in terms of Wigner 3-j symbols, follows from the Phinney & Burridge formalism. The product of the Wigner 3-j symbols is simply related to the integral of three generalized spherical harmonics over the unit sphere (see, e.g., Edmonds 1960). It is worth noting that the appearance of the 1 and -1 in the lower row of the 3-j symbol represented by H_s^1 is related to the gradient in equation (31). This reveals one of the beautiful aspects of the generalized spherical harmonic formalism, that gradients translate simply to index raising in the 3-j symbols.

Another attractive feature of expressing the solution to the forward problem (eq. [33]) in terms of Wigner 3-j symbols is the immediacy of selections rules which result, in part, from properties of the 3-j symbols. Under the isolated multiplet approximation these selection rules are that the frequency perturbation caused by an axisymmetric flow of degree s is nonzero only if (1) the flow is toroidal, (2) s is odd, and (3) $0 \le s \le 2l$. As a consequence, we have written the sum in equation (32) over only odd s. Thus, under the isolated multiplet approximation, only zonal toroidal flows with odd degree less than or equal to twice the degree of the multiplet contribute to the splitting. For the sake of accuracy we should point out that selection rule (1) does not derive from a property of the 3-j symbols. Rather it results from the fact that the integral kernel for a poloidal flow field under the isolated multiplet approximation is identically zero. We do not show this here since we are explicitly considering only differential rotation which is purely toroidal. However, this is demonstrated by Lavely & Ritzwoller (1991).

Equation (32) together with equations (33)–(37) completely specify the forward problem. We have chosen to write equation (32) in a way that has proven useful in geophysical applications (e.g., Ritzwoller, Masters, & Gilbert 1986, 1988; Giardini, Li, and Woodhouse 1988) where the coefficients ${}_nc_{sl}$ would be recognized as splitting function coefficients or as interaction coefficients. We argue that these coefficients, being linearly related to the model parameters w_s^0, are what should be estimated in any analysis of the data aiming to infer differential rotation. We discuss an alternative method of estimating the ${}_nc_{ls}$ coefficients in § 4 and their relation to the splitting coefficients, ${}_na_{li}$ and ${}_nb_{li}$ in § 5.

It is hoped that a major product of this paper will be formulae that are simple and efficient to use both in the forward and inverse problems of differential rotation. The coefficients γ_{sl}^m in equation (33) can be computed numerically and all the results in this paper could be simply stated in terms of them. Numerical methods for computing 3-j symbols are discussed and programs are tabulated in Zare (1988). However, for ease of use we will rewrite the 3-j symbols for $s \le 5$ in terms of polynomials in l and m by using the recursion relation of Schulten & Gordon (1975). If desired, it is straightforward to extend these formulae to $s > 5$ by repeated application of the recursion relation. Setting $j_1 = s$, $j_2 = j_3 = l$, $m_1 = 0$, $m_2 = m$, and $m_3 = -m$ in equation (5a) of Schulten & Gordon (1975) and using equations (36) and (37), we obtain

$$H_{s+1}^m = \frac{1}{s+1}\left\{2(2s+1)mH_s^m - s[(2l+1)^2 - s^2]H_{s-1}^m\right\} . \tag{38}$$

To initiate the recursion, the polynomial forms of the Wigner 3-j symbols appearing on the right-hand side of equation (38) for $s = 1$ are required, and can be found in Table 2 of Edmonds (1960). We find by using equation (38) that for $s \le 5$:

$$H_0^m = 1 , \tag{39}$$

$$H_1^m = 2m , \tag{40}$$

$$H_2^m = 6m^2 - 2L^2 , \tag{41}$$

$$H_3^m = 20m^3 - 4(3L^2 - 1)m , \tag{42}$$

$$H_4^m = 70m^4 - 10(6L^2 - 5)m^2 + 6L^2(L^2 - 2) , \tag{43}$$

$$H_5^m = 252m^5 - 140(2L^2 - 3)m^3 + [20L^2(3L^2 - 10) + 48]m . \tag{44}$$

Algebraic expressions for H_6^m–H_{11}^m have been tabulated in the Appendix. Using equations (39)–(44), γ_{sl}^m for $s \le 5$ can be written:

$$\gamma_{1l}^m = \left(\frac{3}{4\pi}\right)^{1/2} m , \tag{45}$$

$$\gamma_{2l}^m = \left(\frac{5}{4\pi}\right)^{1/2} \frac{(3m^2 - L^2)(3 - L^2)}{4L^2 - 3} , \tag{46}$$

$$\gamma_{3l}^m = \frac{3}{2}\left(\frac{7}{4\pi}\right)^{1/2} \frac{-10m^3 + (6L^2 - 2)m}{4L^2 - 3} , \tag{47}$$

$$\gamma_{4l}^m = \frac{3}{8}\left(\frac{9}{4\pi}\right)^{1/2} \frac{[70m^4 - 10(6L^2 - 5)m^2 + 6L^2(L^2 - 2)](L^2 - 10)}{(4L^2 - 3)(4L^2 - 15)} , \tag{48}$$

$$\gamma_{5l}^m = \frac{15}{16}\left(\frac{11}{4\pi}\right)^{1/2} \frac{252m^5 - 140(2L^2 - 3)m^3 + [20L^2(3L^2 - 10) + 48]m}{(4L^2 - 3)(4L^2 - 15)} . \tag{49}$$

4. ORTHONORMAL BASIS FUNCTIONS FOR REPRESENTING SPLITTING DATA

Observed frequency splittings are typically represented in terms of a Legendre polynomial expansion with argument m/l, where the expansion coefficients are the splitting coefficients ${}_na_{li}$ as in equation (4). (For example, Libbrecht 1989 measured the splitting coefficients of 723 p multiplets in the range $5 \le l \le 60$.) The sum in equation (4) has usually been truncated at $M = 5$ since the inclusion of higher orders does not significantly improve the fit to the data (Brown et al. 1989). Apparently, Legendre polynomials have been chosen to represent the splitting since they are well known basis functions and Legendre functions are orthogonal over the continuous interval $[-1, 1]$:

$$\int_{-1}^{1} P_i(x)P_{i'}(x)dx = \frac{2}{2i+1}\,\delta_{ii'} \,. \tag{50}$$

Legendre functions are not orthonormal, but can easily be orthonormalized. There are two problems with their use. First, they are not an orthogonal basis set for representing discrete data such as split frequencies. If used to represent discrete data they are only approximately "orthogonal." However, the accuracy of this approximation improves with harmonic degree l as the sampling of the interval $[-1, 1]$ becomes finer. Second, and much more significantly, their use complicates the inverse problem. As we will show in § 5 (eqs. [61]–[63]), by representing split frequencies with splitting coefficients ${}_na_{li}$ based on Legendre polynomials, the interaction coefficients ${}_nc_{sl}$ are not related to a single splitting coefficient, but are related to a linear combination of the ${}_na_{li}$.

The purpose of this section is to point out that there exists an orthonormal set of basis functions over the discrete interval $-l \le m \le l$, and that these functions are simply related to the γ_{ls}^m functions. These basis functions are Clebsch-Gordon coefficients. Furthermore, in § 5, we show that the use of Clebsch-Gordon coefficients to represent splitting further simplifies the inverse problem (eq. [56]).

Inspection of equation (32) suggests that a natural, alternative way to represent splitting caused by differential rotation would be to use the functions γ_{sl}^m as basis functions rather than Legendre polynomials. For fixed l, the γ_{sl}^m functions are orthogonal in s. Their orthogonality relation can be deduced from the orthogonality property of Wigner 3-j symbols given by equation (3.7.8) of Edmonds (1960). We rewrite this for our purposes as

$$\sum_{m=-l}^{l} \begin{pmatrix} s & l & l \\ 0 & m & -m \end{pmatrix} \begin{pmatrix} s' & l & l \\ 0 & m & -m \end{pmatrix} = \frac{\delta_{ss'}}{2s+1} \,. \tag{51}$$

Combining this with equations (33) and (36), we obtain the orthogonality relation for the γ_{sl}^m functions:

$$\sum_{m=-l}^{l} \gamma_{sl}^m \gamma_{s'l}^m = \frac{\delta_{ss'}}{2s+1}\frac{G_s G_{s'}}{F_s F_{s'}} \,, \tag{52}$$

where we have defined

$$G_s = \gamma_{sl}^m / H_s^m \,. \tag{53}$$

We note that G_s is independent of m since the m-dependence of γ_{sl}^m is given by H_s^m as can be seen in equation (33).

Since the terms on the right-hand side of equation (52) can vary widely in size with s, especially for high l, the γ_{sl}^m should not be used as basis functions for the splitting. However, equations (51) and (52) suggest a set of basis functions which are orthonormal on the discrete interval $-l \le m \le l$. These orthonormal functions β_{sl}^m are simply related to the γ_{sl}^m functions as follows:

$$\beta_{sl}^m = \frac{(2s+1)^{1/2}F_s}{G_s}\,\gamma_{sl}^m = (2s+1)^{1/2}F_s H_s^m \,, \tag{54}$$

where

$$\sum_{m=-l}^{l} \beta_{sl}^m \beta_{s'l}^m = \delta_{ss'} \,. \tag{55}$$

Equation (54) can be evaluated explicitly in terms of polynomials in m and l for $s \le 5$ by use of equations (39)–(44) and (37). Higher degree β_{sl}^m can be computed by using the formulae provided in the Appendix. By equation (3.7.3) of Edmonds (1960), it can be seen that the β_{sl}^m coefficients are simply Clebsch-Gordon coefficients. Equation (5) would then be used to represent splitting.

5. THE INVERSE PROBLEM

Once the new splitting coefficients ${}_nb_{sl}$ have been estimated using equation (5), the inverse problem for differential rotation can be reconstructed immediately since the ${}_nb_{sl}$ coefficients are simply related to the ${}_nc_{sl}$ coefficients as follows:

$$c_s = \int_0^{R_\odot} w_s^0(r)K_s(r)r^2\,dr = \frac{(4\pi)^{1/2}}{(2l+1)L^2 H_s^1 F_s}\,b_s \,, \tag{56}$$

where we have employed the notational simplification $c_s = {}_nc_{sl}$ and $b_s = {}_nb_{sl}$. Given estimates of the interaction coefficients c_s, equation (56) defines a linear inverse problem for differential rotation.

Although estimating the new splitting coefficients b_s with the orthonormal basis functions β_{ls}^m is more stable than current methods and results in an exceedingly simple inverse problem (eq. [56]), some observers may choose to retain the way splitting data have traditionally been represented. Therefore, we also seek expressions which relate the estimated splitting coefficients ${}_n a_{il}$ to the interaction coefficients ${}_n c_{sl}$. We do this only for $s = 1, 3, 5$ here, although the method we employ can be used to produce higher degree results. First, identify equations (32) and (4):

$$l \sum_{i=1}^{5} a_i P_i(m/l) = \sum_{s=1,3,5} G_s H_s^m c_s , \tag{57}$$

where we have set $a_i = {}_n a_{li}$ and we have defined G_s in equation (53). Then, simply equating terms with the same odd power of m yields:

$$a_1 - \tfrac{3}{2} a_3 + \tfrac{15}{8} a_5 = 2G_1 c_1 + (4 - 12L^2) G_3 c_3 + [20L^2(3L^2 - 10) + 48] G_5 c_5 , \tag{58}$$

$$a_3 - \tfrac{7}{2} a_5 = 8l^2 G_3 c_3 - 56 l^2 (2L^2 - 3) G_5 c_5 , \tag{59}$$

$$a_5 = 32 l^4 G_5 c_5 . \tag{60}$$

Substituting the polynomial representations for γ_{sl}^m (eqs. [45]–[49]) and H_s^m (eqs. [39]–[44]) into G_s, the following identities can be deduced from equations (58)–(60):

$$c_1 = \int_0^{R_\odot} w_1^0(r) K_1(r) r^2\, dr = 2\sqrt{\frac{\pi}{3}} \left[a_1 + \frac{a_3}{2l}\left(3 - \frac{1}{l}\right) + \frac{a_5}{2l}\left(3 + \frac{7}{2l} - \frac{27}{4l^2} + \frac{9}{4l^3}\right)\right] , \tag{61}$$

$$c_3 = \int_0^{R_\odot} w_3^0(r) K_3(r) r^2\, dr = -\sqrt{\frac{\pi}{7}} \frac{(2l-1)(2l+3)}{3l^2} \left[a_3 + \frac{a_5}{2l}\left(7 - \frac{21}{2l}\right)\right] , \tag{62}$$

$$c_5 = \int_0^{R_\odot} w_5^0(r) K_5(r) r^2\, dr = \sqrt{\frac{\pi}{11}} \frac{(2l-3)(2l-1)(2l+3)(2l+5)}{15 l^4} [a_5] . \tag{63}$$

Equations (61)–(63) constitute three independent inverse problems for the radial functions $w_1^0(r)$, $w_3^0(r)$, and $w_5^0(r)$. Equation (63) follows immediately from equation (60). Equation (62) was obtained by substituting equation (63) into equation (59) and solving for c_3. Equation (61) was obtained by substituting equations (63) and (62) into equation (58) and solving for c_1. For clarity, it should be pointed out that the number of terms on the right-hand sides of equations (61)–(63) depends on the accuracy with which the split frequencies are estimated. As frequency estimates become more accurate, the number of terms will increase.

This separation into three independent inverse problems, each uniquely identified with a single harmonic degree s of differential rotation, has been made possible by use of the vector spherical harmonic basis functions given by equations (6) and (12). The corresponding inverse problems for $\bar{\Omega}_k$ and Ω_k do not separate so nicely. The advantage of using the Clebsch-Gordon coefficients as basis functions for splitting is readily apparent by comparing equation (56) with equations (61)–(63). The use of the Clebsch-Gordon coefficients reduces the linear combination on the right-hand side of equations (61)–(63) to a single term in equation (56).

It is beyond the scope of this paper to discuss the large and well-known variety of approaches to linear inverse problems. The reader is directed to the following papers which discuss approaches to these problems in some detail: Backus & Gilbert (1967, 1968, 1970); Parker (1977); Christensen-Dalsgaard et al. (1990). Once $w_1^0(r)$, $w_3^0(r)$, and $w_5^0(r)$ have been estimated from equation (56) or from equations (61)–(63) by whatever inverse method has been chosen, $\bar{\Omega}_k$ and Ω_k for $k = (0, 2, 4)$ can be computed *a posteriori*, if desired, by equations (16)–(18):

$$\Omega_0(r) = \frac{1}{\sqrt{4\pi} r} [\sqrt{3} w_1^0(r) + \sqrt{7} w_3^0(r) + \sqrt{11} w_5^0(r)] , \tag{64}$$

$$\Omega_2(r) = \frac{5}{\sqrt{4\pi} r} [\sqrt{7} w_3^0(r) + \sqrt{11} w_5^0(r)] , \tag{65}$$

$$\Omega_4(r) = \frac{9}{\sqrt{4\pi} r} [\sqrt{11} w_5^0(r)] , \tag{66}$$

$$\bar{\Omega}_0(r) = \frac{1}{\sqrt{4\pi} r} [\sqrt{3} w_1^0(r) - \tfrac{3}{2}\sqrt{7} w_3^0(r) + \tfrac{15}{8}\sqrt{11} w_5^0(r)] , \tag{67}$$

$$\bar{\Omega}_2(r) = \frac{15}{2\sqrt{4\pi} r} [\sqrt{7} w_3^0(r) - \tfrac{7}{2}\sqrt{11} w_5^0] , \tag{68}$$

$$\bar{\Omega}_4(r) = \frac{315}{8\sqrt{4\pi} r} [\sqrt{11} w_5^0(r)] . \tag{69}$$

From equations (61)–(63), it can be easily seen that in the limit of large l, the expansion coefficients $w_1^0(r)$, $w_3^0(r)$, and $w_5^0(r)$ depend only on a_1, a_3, and a_5, respectively. Thus, in the large l limit, these equations can be rewritten:

$$c_1 \approx 2\left(\frac{\pi}{3}\right)^{1/2} a_1 , \tag{70}$$

$$c_3 \approx -\frac{4}{3}\left(\frac{\pi}{7}\right)^{1/2} a_3 , \tag{71}$$

$$c_5 \approx \frac{16}{15}\left(\frac{\pi}{11}\right)^{1/2} a_5 . \tag{72}$$

6. SUMMARY AND CONCLUSIONS

At its inception, the motivation for this paper was that there did not appear to have emerged a fully coherent, unified treatment of the forward and inverse problems for differential rotation. As a result a number of practical problems have beset inversions for differential rotation. In this paper, we present a general formulation of both the forward and inverse problems for differential rotation as well as specific formulae useful to observers with the intent of simplifying solar differential rotation inversion. There are two main points of the paper. (1) If differential rotation is represented by rotational velocity, defined as the zonal, odd degree part of the vector spherical harmonic decomposition of a general convective field in the solar interior, rather than the commonly used rotation rate represented with ad hoc basis functions, then several significant problems currently facing inversions for differential rotation disappear. In particular, the inverse problems for different degrees of rotational velocity are linear and decouple, so that independent inversions for each degree of structure can be performed without approximation. (2) The inverse problems are significantly simplified further if Clebsch-Gordon coefficients are used as basis functions to represent splitting. The Clebsch-Gordon basis functions are genuinely orthonormal on the discrete interval $-l \leq m \leq l$ and regression matrices comprising them are optimally well-conditioned. As a consequence, we highly recommend that the vector spherical harmonic representation of rotational velocity and the Clebsch-Gordon coefficient representation of splitting be adopted to replace the ad hoc representations employed heretofore. We have presented formulae relating the interaction coefficients c_s to both the new splitting coefficients b_s as well as to the traditional splitting coefficients a_s.

Geophysical experience has shown that it is useful to tabulate splitting data in terms of the splitting function or interaction coefficients c_s since these coefficients are linearly related to the structures producing the splitting. However, in the Sun, there are a number of different kinds of axisymmetric asphericities that could produce splitting. In addition to differential rotation, phenomena which have been discussed as potentially large enough to affect p-mode frequencies measurably include lateral density variations caused by large-scale temperature variations, asphericities in the figure of the Sun, and large-scale, dominantly quadrupolar, magnetic fields. Each of these mechanisms affects splitting differently. However, with the possible exception of the poloidal component of magnetic fields (D. Gough, 1990, personal communication), in each case the Wigner-Eckart theorem (Edmonds 1960) guarantees that the solution to the forward problem can be written in a form identical to that of equation (32), but with the γ_{sl}^m coefficients differing in detail for each mechanism of splitting. Thus, an inspection of equation (33) reveals that the interaction coefficients c_s should not be tabulated since they differ in detail among the various sources of splitting. However, the γ_{sl}^m coefficients for the various sources of splitting are similar in that each is linearly related to the same Clebsch-Gordon coefficient. Consequently, we recommend tabulating the new splitting coefficients b_s estimated relative to the Clebsch-Gordon coefficient representation of splitting. For each of the sources of splitting mentioned here, these coefficients are linearly and simply related to the appropriate basis functions representing that mechanism. (For differential rotation we have shown that the appropriate basis functions are given by eq. [12].) Of course, if multiple sources are causing splitting, then models of the various mechanisms would have to be estimated simultaneously.

Throughout the paper, we have attempted to present formulae that would prove to be easy to use. In conclusion, it is worth presenting a brief review of the most important of these. The differential rotation basis function $\boldsymbol{v}_{\rm rot}$ is defined in equations (8) and (12). The general solution to the forward problem is given by equation (32) and the equations immediately following. The specific form of the forward problem for $s \leq 5$ is presented in equations (45)–(49) where polynomials in m, l, and s replace the Wigner 3-j symbols of the general solution. [Formulae useful for calculating the solution to the forward problem analytically for higher degrees ($6 \leq s \leq 11$) are provided in the Appendix.] Equation (34) is the basis for the inverse problem with respect to the ${}_nc_{sl}$ coefficients. The orthonormal basis functions (Clebsch-Gordon coefficients) to represent splitting are defined in equation (54), are tabulated in the Appendix, and the inverse problem with these coefficients is given by equation (56). Relative to the ${}_na_{il}$ coefficients, the inverse problems are given by equations (61)–(63). Once $\boldsymbol{v}_{\rm rot}$ has been estimated, the rotation rate can be computed, if desired, with equations (64)–(69).

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APPENDIX

We have argued in this paper for the following representation of splitting data:

$$\omega_{nl}^{m} = \omega_{nl} + \sum_{s=1}^{M} {}_{n}b_{ls}\beta_{sl}^{m} , \tag{A1}$$

where ${}_{n}b_{ls}$ are the new splitting coefficients and β_{sl}^{m} are Clebsch-Gordon coefficients defined as

$$\beta_{sl}^{m} = (2s + 1)^{1/2} F_s H_s^m , \tag{A2}$$

$$F_s = \left[\frac{(2l - s)!}{(2l + s + 1)!}\right]^{1/2} . \tag{A3}$$

Expressions for H_s^m for $1 \leq s \leq 5$ are given by equations (40)–(44) in the body of the text and for $6 \leq s \leq 11$ below by equations (A4)–(A9). We note that with the H_s^m coefficients provided below, it is straightforward to calculate the γ_{sl}^{m} coefficients for $a > 5$ by use of equation (33).

The H_s^m coefficients for $6 \leq s \leq 11$ are given by

$$H_6^m = 924m^6 - 420m^4(3L^2 - 7) + 84m^2(5L^4 - 25L^2 + 14) - 20L^2(L^4 - 8L^2 + 12) , \tag{A4}$$

$$H_7^m = 3432m^7 - 1848m^5(3L^2 - 10) + 168m^3(15L^4 - 105L^2 + 101) - 8m(35L^6 - 385L^4 + 882L^2 - 180) , \tag{A5}$$

$$\begin{aligned} H_8^m = {} & 12870m^8 - 12012m^6(2L^2 - 9) + 2310m^4(6L^4 - 56L^2 + 81) \\ & -12m^2(210L^6 - 3045L^4 + 9898L^2 - 4566) + 70L^2(L^6 - 20L^4 + 108L^2 - 144) , \end{aligned} \tag{A6}$$

$$\begin{aligned} H_9^m = {} & 48620m^9 - 17160m^7(6L^2 - 35) + 12012m^5(6L^4 - 72L^2 + 145) - 440m^3 \\ & \times(42L^6 - 777L^4 + 3402L^2 - 2630) + 12m(105L^8 - 2660L^6 + 18844L^4 - 36528L^2 + 6720) , \end{aligned} \tag{A7}$$

$$\begin{aligned} H_{10}^m = {} & 184756m^{10} - 145860m^8(3L^2 - 22) + 12012m^6(30L^4 - 450L^2 + 1199) - 2860m^4 \\ & \times(42L^6 - 966L^4 + 5481L^2 - 6248) + 132m^2(105L^8 - 3290L^6 + 29680L^4 - 78900L^2 + 32208) \\ & -252L^2(L^8 - 40L^6 + 508L^4 - 2304L^2 + 2880) , \end{aligned} \tag{A8}$$

$$\begin{aligned} H_{11}^m = {} & 705432m^{11} - 1847560m^9(L^2 - 9) + 58344m^7(30L^2 - 550L^2 + 1869) - 120120m^5 \\ & \times(6L^6 - 168L^4 + 1199L^2 - 1873) + 1144m^3(105L^8 - 3990L^6 + 44730L^4 - 156200L^2 + 105228) \\ & -24m(231L^{10} - 11165L^8 + 174328L^6 - 1006764L^4 + 1771440L^2 - 302400) . \end{aligned} \tag{A9}$$

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