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Feasibility of Truncated Perturbation Expansions to Approximate Rayleigh-Wave Eigenfrequencies and Eigenfunctions in Heterogeneous Media

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Abstract We investigate the feasibility of using truncated perturbation expansions higher than first order to compute the effect of structural variations relative to a laterally homogeneous reference on broadband (e.g., 20 to 250 sec period) Rayleigh-wave velocities and eigenfunctions. Feasibility is a function of speed, accuracy, and ease-of-use. We discuss the physical meaning of relevant terms in the expansion and posit and test an expansion, referred to as quasi-third-order theory, that consists of all boundary and volume self-terms through third-order and at second-order boundary-volume cross-terms between all boundaries and adjoining volumes that the boundary intersects. We set accuracy criteria at 0.5% for group and phase velocities and several percent for vertical eigenfunctions. For the magnitude of crustal and upper mantle heterogeneities found across Eurasia, first-order perturbation theory meets these criteria for group and phase velocities only above about 80 sec period but meets the eigenfunction criterion down to about 30 sec period. The use of quasi-third-order theory for phase and group velocities and the first-order theory for eigenfunctions is fast (about two orders of magnitude faster than the flat-earth and spherical-earth eigenfunctions codes used for comparison), relatively easy to use, and should meet the accuracy criteria required in most inversions down to about 30 sec period. If accuracy standards are more stringent than those set here, if there are structural variations larger than those considered here, or if the application requires inversion below about 30 sec period, then it would be advisable to regionalize the area of study and to introduce more than one reference model.

Introduction

It is common in large-scale seismology to treat lateral variations in the structure of the Earth as perturbations to a global or large-scale reference. Because of its computational efficiency, a Taylor series expansion truncated at first order (we refer to this as first-order perturbation theory) is frequently used to compute the effect of these structural variations on seismic observables such as surface-wave group and phase velocities and normal-mode frequencies. The efficiency of this theory results from the fact that it only requires a single computation of the partial derivative of each observable with respect to each of the structural variables, and these derivatives can be computed *a priori* and tabulated. Perturbation expansions are much faster, therefore, than solving for eigenfrequencies and eigenfunctions directly. It is, however, generally recognized that the use of first-order perturbation theory below about 100 sec period incurs significant errors if the deviations from the reference state are large. In particular, it is well known (but not conspicuously documented) that topography on the Moho discontinuity notoriously causes the first-order theory to break down.

The breakdown of the first-order theory below 100 sec period is a serious problem for broadband surface-wave inversions across large and diverse geographical areas. A considerable computational price must be paid to compute dispersion curves and eigenfunctions on a grid across the entire region of study. The magnitude of this problem scales with the size of the studied region and inversely with the grid spacing. The goal of this article is to consider higher-order perturbation theories and to investigate the feasibility of their use in broadband surface-wave inversions, where feasibility is a function of speed, accuracy, and ease-of-use.

There are several caveats concerning the generality of the results presented here. First, the results depend in detail on model parameterization. For simplicity of use and interpretation, we consider structural perturbations to be constant in layers. Second, the results also depend in detail on the nature of the reference model, not just on the type of the basis functions. In particular, for a layered model, there is a trade-off between accuracy and ease-of-use with respect to layer thicknesses. Finally, the conclusions reached about the feasibility of the use of higher order perturbation theories

are dependent on the desired accuracy and the size of the heterogeneities relative to the reference.

The accuracy standards that we set are the following: 0.5% in phase and group velocities below 100 sec period, somewhat better at longer periods, and several percent (i.e., ~5%) for eigenfunctions. We take Earth structures in the crust and mantle from the models CRUST-5.1 of Mooney *et al.* (1998) and S16B30 of Masters *et al.* (1996). These structures are then represented as perturbations relative to a reference model that approximates the average velocity structure under stable platform regions in Asia. This continent-wide reference, which we call ESC (Eurasian Stable Continent), is based on the stable Eurasian model of Lerner-Lam and Jordan (1983). ESC is composed of 10 homogeneous spherical shells and 4 first-order discontinuities from the solid surface to 400 km depth and is underlain by PREM (Dziewonski and Anderson, 1981). It, thus, comprises 14 model parameters on 51 radial knots. We present accuracy results at four structurally diverse geographical locations: the Northern Caspian depression (thick sediments), the Siberian shield (very similar to ESC), Northern Japan (thin crust), and Southern Tibet (thick crust). The accuracy of the perturbation theories relative to a global average, such as PREM, would naturally be much worse.

The next section presents a discussion of the perturbation expansion, including definitions and discussion of the physical meaning of the higher order terms. Based on the physical insight gained from these cases, we adopt, for further test, an approximation to higher order perturbation theory that is considerably more accurate than first-order theory but remains relatively easy to use. We refer to this approximation as a "quasi-third-order perturbation theory." Then we present tests of this expansion by investigating the accuracy of this theory applied at the four chosen points around Eurasia.

Perturbation Expansions

Consider a spherically symmetric, isotropic, anelastic reference model defined by the vector $m_o(r) = [\beta(r), \alpha(r), \rho(r), Q(r)]^T$, where r is radius, β is shear velocity, α is compressional velocity, ρ is density, and Q is the quality factor. Isotropic, location-dependent perturbations to the reference are given by $\delta\mathbf{m}(r) = [\delta\beta(r), \delta\alpha(r), \delta\rho(r), \delta Q(r)]^T$. Let $\mathbf{m}(r)$ denote the sum of the reference and the perturbation. For simplicity, assume that $\delta\mathbf{m}(r)$ is constant radially in a set of N spherical shells and that there are also perturbations, h_1, h_2, . . . , h_m, to the radii of a set of M discontinuities that exist in the reference model $\mathbf{m}_o$. With these definitions, first-order perturbation theory for group (U) and phase (c) velocities and the vertical or horizontal radial eigenfunctions (ξ) is simply a truncated Taylor Series expansion defined as follows:

$$U(\omega) \approx U_0(\omega) + \sum_{n=1}^{N} \left(\frac{\partial U(\omega)}{\partial \mathbf{m}_n}\right) \cdot \delta\mathbf{m}_n + \sum_{m=1}^{M} \left(\frac{\partial U(\omega)}{\delta h_m}\right) h_m, \quad (1)$$

$$c(\omega) \approx c_0(\omega) + \sum_{n=1}^{N} \left(\frac{\partial c(\omega)}{\partial \mathbf{m}_n}\right) \cdot \delta\mathbf{m}_n + \sum_{m=1}^{M} \left(\frac{\partial c(\omega)}{\delta h_m}\right) h_m, \quad (2)$$

$$\xi(r, \omega) \approx \xi_0(r, \omega) + \sum_{n=1}^{N} \left(\frac{\partial \xi(r, \omega)}{\partial \mathbf{m}_n}\right) \cdot \delta\mathbf{m}_n + \sum_{m=1}^{M} \left(\frac{\partial \xi(r, \omega)}{\delta h_m}\right) h_m, \quad (3)$$

where $U_o(\omega)$ is the group velocity curve, $c_o(\omega)$ is the phase velocity curve, and $\xi(r)$ is the frequency-dependent radial eigenfunction (either horizontal or vertical) for $\mathbf{m}_o$, $\delta\mathbf{m}_n$ is the volumetric perturbation in layer n, and h_m is topography on discontinuity m. Hereafter, equations will be presented only for phase velocity because the group velocity and radial eigenfunction expressions are identical in mathematical form. We compute partial derivatives using finite differences and the spherical Earth eigenfunction code of Woodhouse (1988).

Because Rayleigh-wave velocities are dominantly controlled by shear velocities, we will consider volumetric perturbations only to shear velocity and assume that α, ρ, and Q are position independent. The results of the tests we report here are not affected appreciably by this assumption. Thus, the vector $\delta\mathbf{m}$ hereafter will be considered to be a scalar, δm, to be thought of as a shear velocity perturbation.

The accuracy of the first-order theory to predict group and phase velocities at the four geographical locations is presented in Figure 1. Phase velocities are generally more accurately predicted than group velocities. Because the local models differ most appreciably from the reference model ESC under Japan and Tibet, the first-order theory is least accurate in these locations. With a group velocity accuracy standard of about 0.5%, first-order perturbation theory breaks down at these locations relative to ESC at periods below about 80 to 100 sec, consistent with the commonly held beliefs about the theory.

A general perturbation theoretic expansion to order P is defined exactly as a higher-order Taylor Series expansion (e.g., Arfken, 1985; sec. 5.6):

$$c(\omega) \approx c_0(\omega) + \sum_{p=1}^{P} \frac{1}{p!} \left\{ \sum_{n=1}^{N} \delta m_n \left(\frac{\partial}{\partial m_n}\right) + \sum_{m=1}^{M} h_m \left(\frac{\partial}{\partial h_m}\right) \right\}^p c(\omega). \quad (4)$$

This expression has been written in operator notation where,

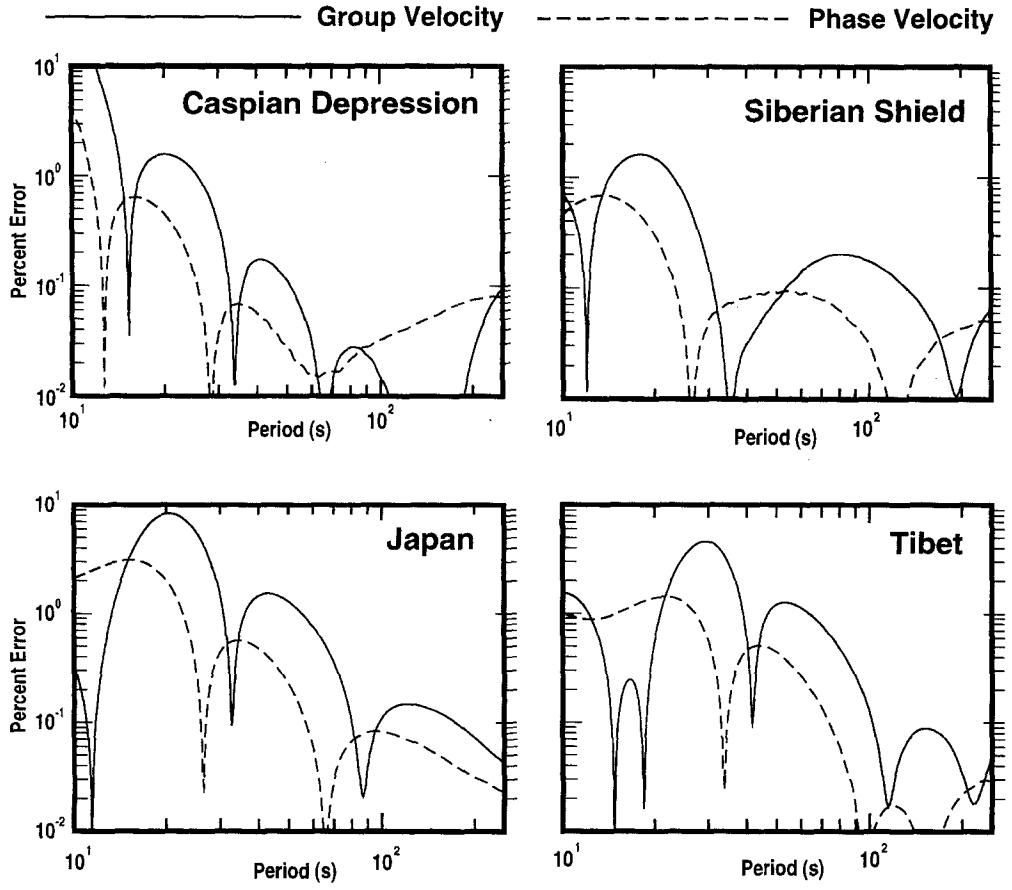

Figure 1. Accuracy of the group (solid line) and phase (dashed line) velocity curves for four diverse geographical locations computed with first-order perturbation theory. Accuracy is defined as the percentage difference between the perturbation-theoretic curve and the spherical-earth eigenfunction curve (Woodhouse, 1988) at each period.

for example, the term $(\partial/\partial h_m)^P$ should be understood as $\partial^p/\partial h_m^p$. We aim to identify the most significant terms in the expansion in order to define a simplified expansion that approximates the full expansion well. Discussion is aided by the introduction of some terminology.

There are five types of terms in equation (4) in two general categories that we refer to as "self" and "cross" terms. "Self-terms" involve derivatives of only a single model parameter. Thus, $\partial^2 c/\partial m_n^2$ is a second-order self-term with respect to volumetric structure in the nth layer. "Cross-terms" involve derivatives with respect to more than one model parameter. For example, $\partial^2 c/\partial m_n \partial h_m$ is a second-order boundary-volume cross-term with respect to volumetric structure in the nth layer and topography on the mth discontinuity. There are also cross-terms between volumes (e.g., $\partial^2 c/\partial m_n \partial m_k$) and between boundaries (e.g., $\partial^2 c/\partial h_m \partial h_k$). Thus, the five types of terms at order p are volumetric-self $[\mathbf{V}^{(p)}]$, boundary-self $[\mathbf{B}^{(p)}]$, boundary-volume-cross $[\mathbf{BV}^{(p)}]$, volume-volume-cross $[\mathbf{VV}^{(p)}]$, and boundary-boundary-cross $[\mathbf{BB}^{(p)}]$. With this definition of these operators, equation (4) can be rewritten as follows:

$$c(\omega) \approx c_0(\omega) + \sum_{p=1}^{P} \frac{1}{p!} \{\mathbf{V}^{(p)} + \mathbf{B}^{(p)} + \mathbf{BV}^{(p)} + \mathbf{VV}^{(p)} + \mathbf{BB}^{(p)}\}\, c(\omega). \quad (5)$$

All cross-terms are zero at first-order: $\mathbf{BV}^{(1)} = \mathbf{VV}^{(1)} = \mathbf{BB}^{(1)} = 0$. Note that if perturbations $\delta\alpha$, $\delta\rho$, and δQ were also included, equation (5) would be complicated further by the existence of volume–volume cross-terms between structures of different types, for example, shear velocity–compressional velocity cross-terms. These terms are generally too small to consider further. The methods used to compute the partial derivatives numerically and a discussion of numerical stability are presented in James (1998).

The only significant boundary–volume cross-terms are those between each boundary and the volumetric layers that topography on that boundary intersects. Thus, only a small subset of the terms that define $\mathbf{BV}^{(p)}$ are significant. The physical significance of an expansion that retains only volume and boundary self-terms and volume–boundary cross-terms between boundaries and adjoining volumes can be understood by inspecting Figure 2. Consider a model composed of two layers: an upper layer (layer 1) with unper-

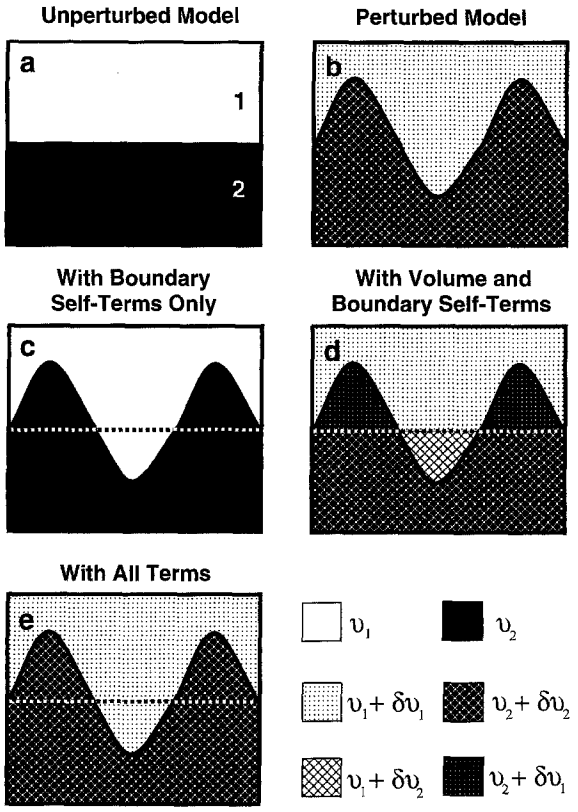

Figure 2. Illustration of the effect of retaining different terms in the perturbation expansion for a simple two-layer model. (a) The unperturbed and (b) the perturbed models. (c) Boundary self-terms represent moving the boundary without modifying volumetric structure. (d) The addition of volume self-terms changes the velocity in each layer, but errors in the hills and valleys remain. (e) Inclusion of boundary–volume cross-terms corrects the velocities in the hills and valleys.

turbed velocity v_1 and a lower layer (layer 2) with unperturbed velocity v_2 (Fig. 2a) separated by a boundary at radius r. Introduce topography on the boundary between the layers and a perturbation in velocity in both layers such that $r \rightarrow r + h$, $v_1 \rightarrow v_1 + \delta v_1$, and $v_2 \rightarrow v_2 + \delta v_2$ (Fig. 2b). The variables h, v_1, and v_2 may vary horizontally. The boundary self-terms represent the effect of perturbing the boundary with no perturbation in the velocities (Fig. 2c). The volume self-terms include the effect of perturbing the velocities in both volumes, but the velocities in the hills and valleys remain incorrect (Fig. 2d). The hills should have velocity $v_2 + \delta v_2$, but are approximately $v_2 + \delta v_1$ because they fall in the radial layer that originally was part of layer 1, and, therefore, the layer 1 velocity perturbation has been applied to them by the volume self-terms that are not cognizant of topography. Similarly, the valleys should be $v_1 + \delta v_1$ but are approximately $v_1 + \delta v_2$. The cross-terms between the boundary and layers 1 and 2 act to correct this error by approximately replacing the erroneous velocities in the hills and valleys with the correct perturbed velocities: $v_1 + \delta v_2 \rightarrow\sim (v_1 + \delta v_1)$ and $v_2 + \delta v_1 \rightarrow\sim (v_2 + \delta v_2)$.

For a heuristic understanding of how the boundary–volume cross-terms replace the erroneous velocities in the hills and valleys with more accurate velocities, consider the second-order boundary–volume cross-terms in this simple model:

$$\mathbf{BV}^{(2)} = 2\left(\frac{\partial^2 c}{\partial v_1 \partial h} h\delta v_1 + \frac{\partial^2 c}{\partial v_2 \partial h} h\delta v_2\right) \quad (6)$$

$$\approx \frac{\partial^2 c}{\partial v_1 \partial h} h(\delta v_1 - \delta v_2) + \frac{\partial^2 c}{\partial v_2 \partial h} h(\delta v_2 - \delta v_1), \quad (7)$$

where the latter approximate equality follows because $\partial^2 c/\partial v_2 \partial h \approx -\partial^2 c/\partial v_1 \partial h$. The second-order boundary–volume cross-terms add to the expansion a term proportional to $\delta v_2 - \delta v_1$, which is precisely what is needed to convert $v_2 + \delta v_1$ to $v_2 + \delta v_2$, and a term proportional to $\delta v_1 - \delta v_2$, which is needed to convert $v_1 + \delta v_2$ to $v_1 + \delta v_1$. The near antisymmetry between the second-order boundary–volume cross-terms for the layers straddling the Moho in ESC (i.e., layers 4 and 5) results from the fact that a positive velocity perturbation in layer 5 increases the velocity jump on the Moho relative to ESC, whereas a positive perturbation in layer 4 decreases the jump. James (1998) provides further documentation of this antisymmetry.

Figure 3 presents accuracy estimates for the 50 sec Rayleigh wave for six different simple models. All models are taken relative to ESC. The first model (Fig. 3a) contains only a +10-km perturbation on the Moho. Boundary self-terms are the only nonzero terms in the expansion for this model, and the series converges to better than 0.5% accuracy by third order. The second model (Fig. 3b) contains only a 10% shear velocity perturbation to layer 5 in ESC, which is the layer directly underlying the Moho. Convergence occurs by second order. In general, convergence occurs at lower order for realistically large perturbations in volumetric structure than for large topographic perturbations. Cross-terms come into play when more than one model parameter perturbation is included. Volumetric perturbations to layers 4 and 5, straddling the Moho, are included in the third model (Fig. 3c), but there is no topography on the Moho. Volume–volume terms are the only nonzero cross-terms for this model. These terms, like boundary–boundary cross-terms, are smaller than the self-terms and the largest boundary–volume cross-terms at every order. Hence, we drop $\mathbf{VV}^{(p)}$ and $\mathbf{BB}^{(p)}$ from further consideration. The fourth model (Fig. 3d) has +10 km of topography on the Moho and a +10% shear velocity perturbation in layer 5 directly underlying the Moho. Boundary and volume self-terms alone yield about a 1% error in group velocity, which is corrected with the addition of the boundary–volume cross-term. Convergence to about 0.5% error occurs by second order, but an appreciable correction is applied by the third-order boundary–volume

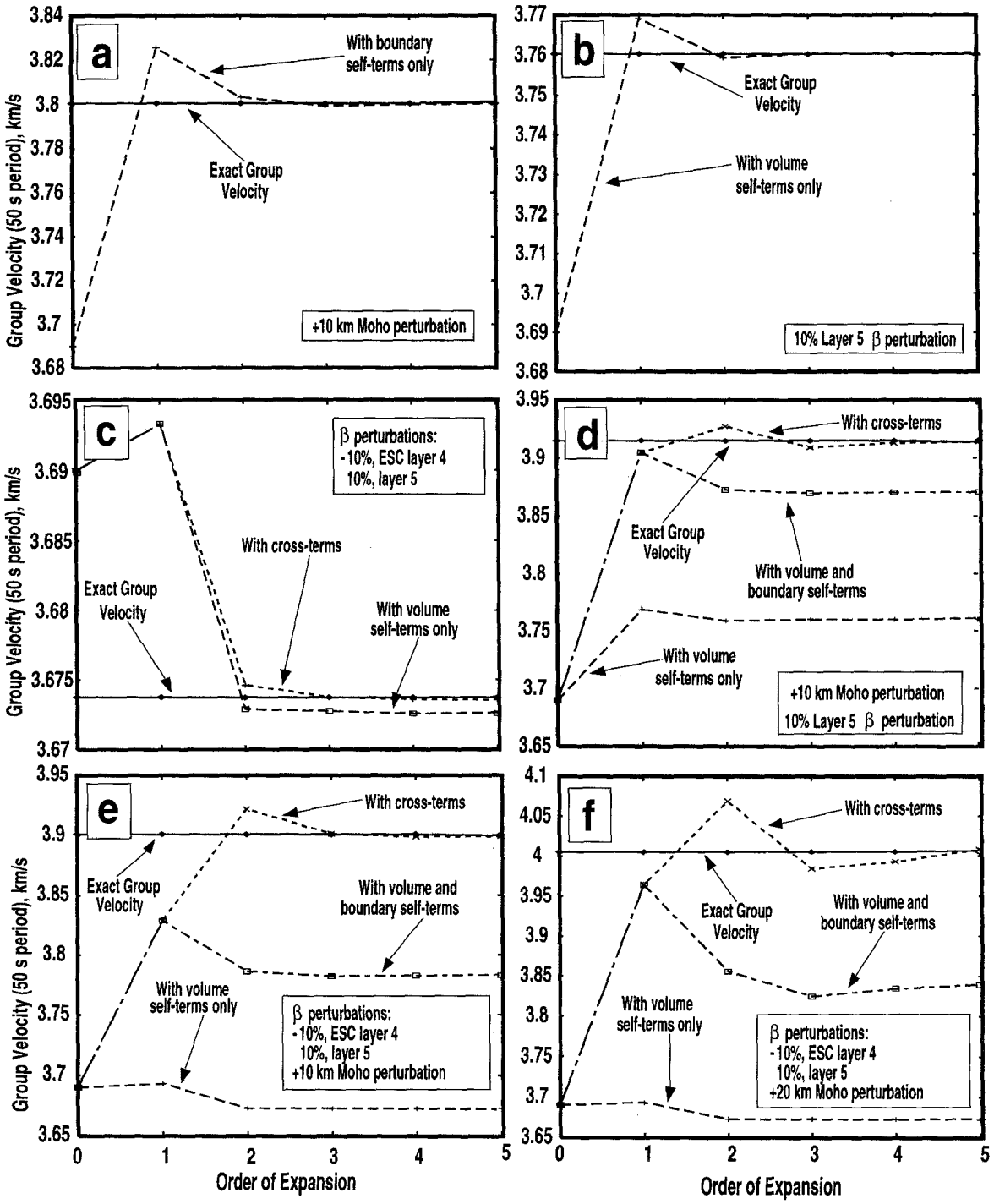

Figure 3. Illustration of the effect on the group velocity of the 50 sec Rayleigh wave of retaining higher order terms in the perturbation expansion for six simple models. The correct velocity is shown with the horizontal line. The zeroth-order velocity is for the reference model ESC (~3.69 km/sec). (a) +10-km Moho perturbation; that is, crust is thinned by 10 km. (b) +10% perturbation in shear velocity in layer 5 of ESC, directly below the Moho. (c) +10% perturbation in layer 5 and −10% perturbation in shear velocity in layer 4 of ESC, directly above the Moho. The effect of volume–volume cross-terms is shown. (d) +10-km Moho perturbation and 10% perturbation in shear velocity in layer 5. The effect of adding boundary self-terms and boundary–volume cross-terms is shown. (e) +10-km Moho perturbation, 10% perturbation in shear velocity in layer 5, and −10% perturbation in layer 4. The effect of adding boundary self-terms and boundary–volume cross-terms is shown. (f) Same as (e), except with +20 km of topography on the Moho.

term. Models five and six have -10% and $+10\%$ perturbations in shear velocity in layers 4 and 5, respectively, and $+10$ km (Fig. 3e) or $+20$ km (Fig. 3f) of topography on the Moho. These figures clearly indicate the importance of the cross-terms between topography on the Moho and its adjoining volumetric layers. Errors in excess of 3% can occur when these terms are neglected. With $+10$ km on the Moho, convergence occurs by third order, but fourth-order terms are required to achieve 0.5% errors with $+20$ km on the Moho for the 50 sec Rayleigh wave.

Based on the results presented in Figure 3, we posit the following simplification as a computationally feasible approximation to the full perturbation expansion. This approximation retains all self-terms through third-order and second-order boundary–volume cross-terms between all boundaries and adjoining volumetric layers that the topography intersects. Mathematically, this simplification to equation (5) can be written as follows:

$$c(\omega) \approx c_0(\omega) + \left\{ \mathcal{B}V^{(2)} + \sum_{p=1}^{3} \frac{1}{p!} (\mathbf{V}^{(p)} + \mathbf{B}^{(p)}) \right\} c(\omega), \quad (8)$$

where $\mathcal{B}V^{(2)}$ denotes only those cross-terms in $\mathbf{BV}^{(2)}$ between topography on each boundary and the volumetric layers that the topography intersects. We call the expansion in equation (8) a "quasi-third-order" theory because some of the largest third-order terms are retained. Of course, because volume–volume cross-terms, boundary–boundary cross-terms, and even some boundary–volume cross-terms are also not included, it is not even a full second-order theory. The number of terms in this expansion is 55 compared with 671 in the full third-order theory. If topography causes boundaries to narrowly approach one another as layers pinch off, then other terms should be retained, in particular, boundary–boundary cross-terms.

Test of the Quasi-Third-Order Theory

Figure 4 presents the results of a test of the accuracy of the quasi-third-order theory represented by equation (8), evaluated at four geographical locations. Comparison with Figure 1 shows that, like the first-order theory, quasi-third-order theory is most accurate at long periods and at locations that are structurally similar to ESC (that is, Caspian Depression, Siberian Shield), and phase velocities are predicted slightly better than group velocities on average. However, the quasi-third-order theory provides a significant improvement over the first-order theory. The quasi-third-order theory meets the accuracy standard of approximately 0.5% down to about 30 sec period at all four sites. The biggest problems for the theory are at and below 20 sec period at all sites and at Tibet and Japan because these regions are most unlike ESC.

The problems for the quasi-third-order theory are largely attributable to perturbations to crustal thickness (Moho topography plus free-surface topography). For example, the model crust under Japan is about 27 km thick compared to 43 km for ESC, and the model crust under Tibet is about 57.5 km thick. This is documented further in James (1998).

In summary, quasi-third-order perturbation theory can be used to compute group and phase velocities down to about 30 sec period with an accuracy of better than about 0.5% and with up to about ± 15 km perturbation in crustal thickness relative to the reference model. If this accuracy is sufficient, the theory should be able to be applied across most continents with a single reference model. Across regions with larger structural perturbations to the crust (e.g., continent–ocean variations), more than one reference model would have to be used to retain this level of accuracy or the theory would have to be applied only at somewhat longer periods, say at periods above 40 to 50 sec.

To this point, we have not yet discussed the accuracy of eigenfunctions. An accuracy of $\sim 5\%$ for the eigenfunctions is sufficient for most purposes because errors involved in using eigenfunctions are usually somewhat larger than this (e.g., instrument responses, source depth, theoretical errors in synthetic formalisms, etc.). We concluded above that phase and group velocities predicted by the first-order perturbation theory meet the accuracy criterion of 0.5% in general only above about 80 to 100 sec period. Figure 5 compares first-order perturbation-theoretic eigenfunctions with those from ESC and those from the local models for Tibet and Japan. Because the perturbation-theoretic eigenfunctions are evaluated on the radial knots of the model ESC, some details of the local eigenfunctions are difficult to reproduce. However, the first-order perturbation-theoretic vertical eigenfunctions are generally accurate at the $\pm 3\%$ level down to about 30 sec period, the short-period boundary above which quasi-third-order perturbation theory met the accuracy criterion for group and phase velocities. Thus, first-order eigenfunctions are sufficiently accurate to be used with the quasi-third-order group and phase velocities. If more accurate eigenfunctions are desired, particularly at periods below 30 sec, then the second-order terms of the quasi-third-order expansion should be retained for the eigenfunctions as well.

Conclusions

This study has been motivated by the desire to improve the speed of inversions of broadband surface-wave dispersion curves across continents. Because structural partial derivatives in inversion codes are usually computed numerically, this speed up depends largely on the ability to accelerate the solution of the forward problem. In addition, model space sampling methods such as Monte Carlo methods, genetic and evolutionary algorithms, etc., depend on solving the forward problem efficiently and often. Thus, a

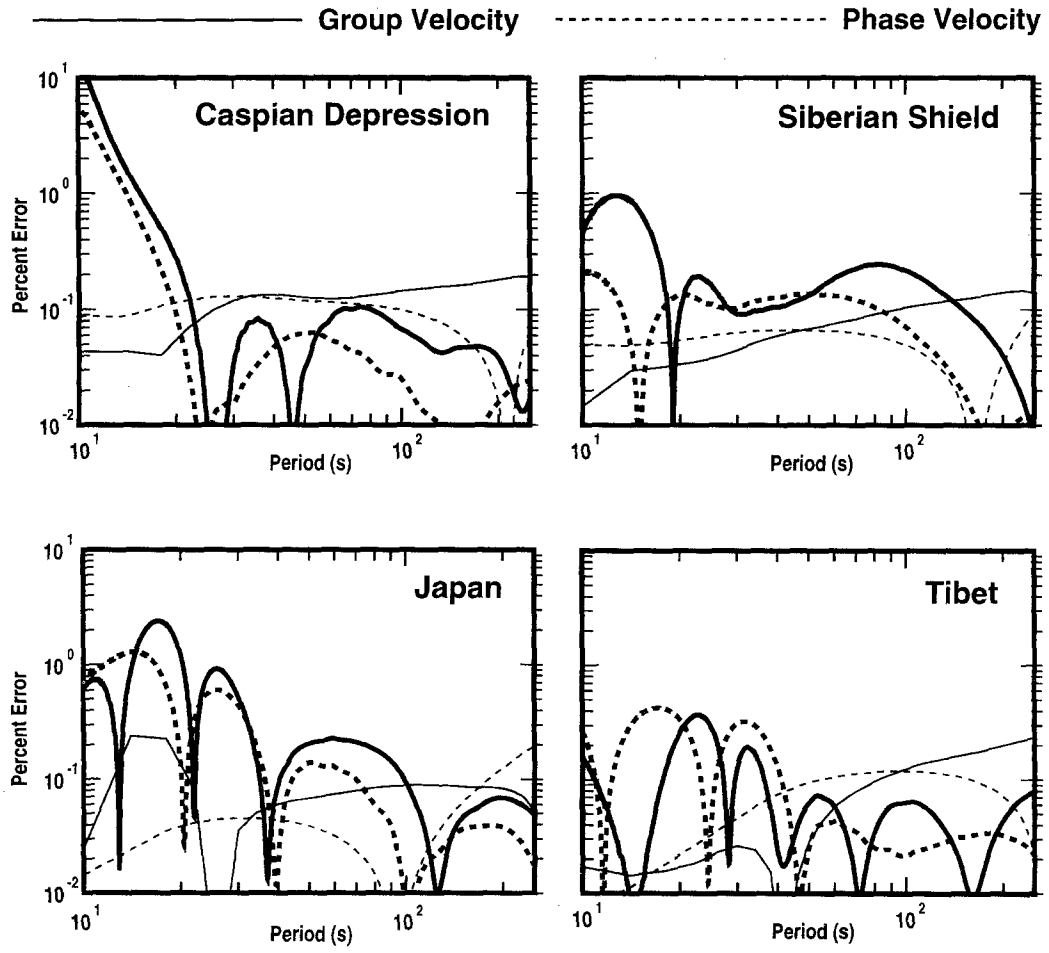

Figure 4. Accuracy of quasi-third-order perturbation theory for group velocity (bold solid line) and phase velocity (bold dashed line) for four geographical locations. The accuracy of the flat-earth eigenfunction code with an earth-flattening transformation is shown for comparison (group velocity, thin solid line; phase velocity, thin dashed line). Accuracy is defined as in Figure 1.

large reduction in the time needed to solve the surface-wave forward problem may not only result in models being constructed faster but also, perhaps, in better characterized and understood models.

We estimate that the accuracy needed in most surface-wave inversions is no more than about 0.5% for group and phase velocities at periods below 100 sec but is somewhat smaller than this at longer periods. Eigenfunctions can display significantly degraded accuracy, with accuracies of several percent (i.e., ~5%) sufficient for most applications.

We have shown that for structural perturbations across continents, first-order perturbation theory meets the accuracy criterion for group and phase velocities above about 80 to 100 sec period. However, below about 80 sec period, the first-order theory breaks down in regions, such as Japan and Tibet, that differ appreciably from the reference model ESC. Because the accuracy criterion for eigenfunctions is more liberal, first-order perturbation-theoretic eigenfunctions meet the accuracy criterion down to a period of about 30 sec, even for Japan and Tibet.

To apply perturbation theories with confidence below about 80 sec period across continents, higher order terms in the perturbation expansion must be retained. Because the number of terms proliferates rapidly with the order of the expansion and the computation of the partial derivatives is relatively unstable, it is necessary to choose the terms included in the expansion at orders beyond the first with care. Fortunately, the vast majority of terms in the expansion are insignificant. The most significant terms in the expansion are volume and boundary self-terms and cross-terms between boundary topography and volumetric structures adjacent to the boundary. By retaining self-terms through third-order and second-order boundary–volume cross-terms between all boundaries and adjoining volumes that the topography intersects, the number of terms in the expansion is decreased by more than an order-of-magnitude relative to the full third-order theory (671 → 55), but the reduced expansion is nearly as accurate as the full third-order theory. We call this reduced expansion "quasi-third-order perturbation theory,"

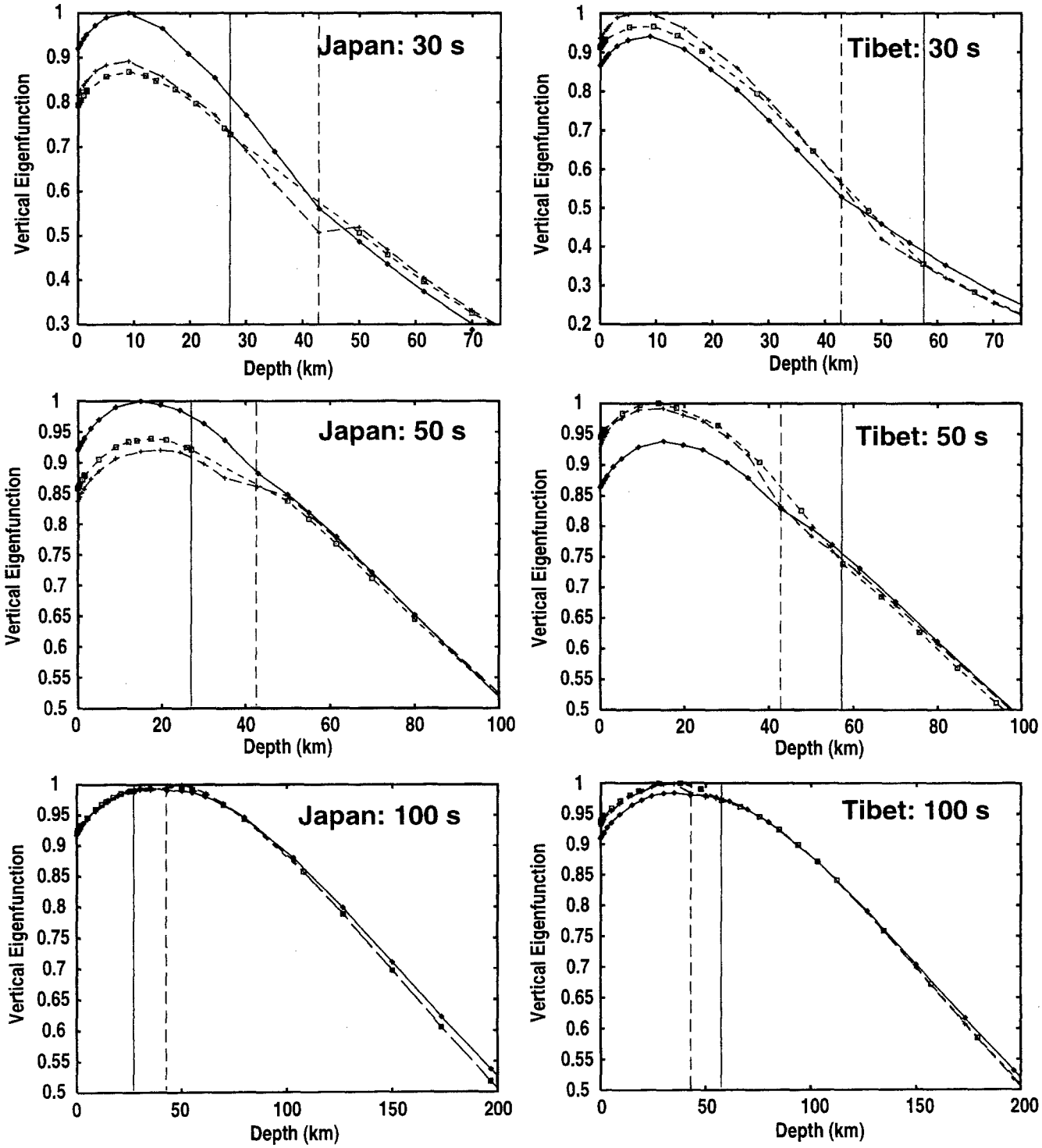

Figure 5. Comparison of three sets of eigenfunctions at three periods (30, 50, and 100 sec) for the models of Tibet and Japan. The solid line with diamonds is for ESC, the dashed line with pluses is for first-order perturbation theory, and the dotted line with squares is the correct eigenfunction. The vertical lines represent crustal thickness in ESC (42.9 km) and in the local model. Eigenfunctions have been normalized on each plot but have been normalized identically in each plot so that comparison can be made meaningfully.

which we find meets the group and phase velocity criteria across Eurasia down to about 30 sec period.

Quasi-third-order perturbation theory is not as accurate as spherical-earth eigenfunction codes or flat-earth codes with appropriate earth-flattening transformations, but it is much faster. Our quasi-third-order code is more than 100 times faster than the flat-earth code we use for comparison (Herrmann, 1978), which is itself somewhat faster than the spherical-earth code (Woodhouse, 1988). This speed increase of two orders of magnitude over the full solutions in flat or spherical geometries translates into a similar improvement in the speed of the inversion.

In conclusion, because of the number of terms, the use of perturbation theories beyond first order is tedious, and the computation of accurate partial derivatives can be complicated. It is possible, however, to strike a balance between speed and accuracy and, by retaining only selected terms in the expansion, to produce an expansion that is fast, relatively easy to use, and meets the accuracy criteria needed in inversions. If accuracy standards more stringent than those set here are required, if there are structural variations larger than those considered here, or if periods below about 30 sec are required in the inversion, then it would be advisable to regionalize the area of study and introduce more than one reference model.

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